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An algorithmic approach towards the tracing procedure of Harsanyi and Selten *

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Abstract

Harsanyi and Selten (1988) proposed an equilibrium selection theory for N-person noncooperative games. Their main subroutine for selecting the appropriate equilibrium is the tracing procedure which converts any given prior into an equilibrium. In this paper we show that the linear tracing procedure is equivalent, up to projection, to a pivoting procedure presented by the authors. As a consequence of this equivalence we derive that the linear tracing procedure always generates a perfect equilibrium whenever it is started from a completely mixed prior. Our procedure also works in some cases in which the linear tracing procedure is not well-defined. Furthermore, it has the additional advantage that it is easily implementable on a computer when applied to bi-matrix games.

Keywords: Tracing procedure, equilibrium selection, complementary pivoting, expanding set, computation.

1 Introduction

The tracing procedure of Harsanyi and Selten (1988) is a well-known method used for selecting a Nash equilibrium in an N-person noncooperative game. In fact, it is part of a more complex equilibrium selection procedure that can be applied to noncooperative extensive and normal form games, as well as to cooperative games. Within this procedure the tracing procedure is used for selecting an equilibrium in certain well-defined subgames. These are games with restricted sets of players and strategies, that are closed with respect to best replies. In the sequel we only consider such well-defined games in strategic form.

The tracing procedure starts from some common prior, i.e., the ideas about the strategy used by any player are the same for all other players. Then the players react optimally to these expectations. In general they observe that their expectations are not fulfilled and thus adjust their expectations about the behaviour of the other players. By simultaneously and gradually adjusting the expectations by the acquisition of more and more information, and reacting optimally against these revised expectations, eventually an equilibrium is reached.

In van den Elzen and Talman (1991), a pivoting method for finding Nash equilibria in bi-matrix games is proposed. The interpretation of the process given there is as follows. The players start from playing a given (mixed) strategy. By adjusting their strategies and by taking into account the starting strategy vector, an equilibrium is reached. More precisely, the probabilities with which the optimal actions are played are increased, whereas all other probabilities are decreased. The starting vector keeps on playing an important role during the whole adjustment process since adjustments are made relative to the initial strategy vector. This method can easily be implemented on a computer because it is a complementary pivoting algorithm. It can also be generalized for games with more than two players. However, we then have to work with a system of nonlinear equations and computations become cumbersome.

Thus, in both procedures the initial strategy vector is important during the process. We will show that, for most priors, the paths generated by both procedures are the same. We do this by revealing that both methods are of the homotopy type. The pivoting method can be seen as tracing a path related to a homotopy with respect to the strategy set, i.e., it generates a sequence of restricted Nash equilibria where the strategies are restricted to lie in an expanding set around the prior. On the other hand, the tracing procedure follows a path of Nash equilibria related to a homotopy with respect to the payoffs. Initially these payoffs are only dependent on the prior, whereas they are gradually converted into the original payoffs of the game. It turns out that there is a 1-1 relation between the vectors on both paths.

The tracing procedure has two variants; the linear tracing procedure and the logarithmic tracing procedure. The latter one is always well-defined and leads to a unique equilibrium. However, the logarithmic tracing procedure is difficult to handle in practice. This because the paths involved are nonlinear. Given a fixed game, the linear tracing procedure is well-defined for almost all priors. In those cases its path is equal to that of the logarithmic tracing procedure. For a given game, the linear tracing procedure is not well-defined in two cases. Firstly, if the optimal reply against the prior is not unique, and secondly for a lower-dimensional set of priors from which the path generated reveals a bifurcation. The process of van den Elzen and Talman (1991) is not well-defined in a strict subset of these cases. We will see later on that one type of bifurcation can be handled. In case of two players the process of van den Elzen and Talman turns out to be a complementary pivoting procedure in a system of linear equations. In that case we can resolve the degeneracy problems by applying lexicographic pivoting or Bland's rule (see Bland (1977)). This would result in an implementable alternative for the logaritmic tracing procedure.

From the equivalence of the linear tracing procedure and the pivoting method we derive that the linear tracing procedure generates a perfect equilibrium whenever that procedure is well-defined and the prior is completely mixed. Thus, in such a case we do not have to apply the linear tracing procedure to a sequence of perturbed games, which is usually needed to obtain perfectness.

The set-up of this paper is as follows. In Section 2 we shortly review the tracing procedure of Harsanyi and Selten and the pivoting method of van den Elzen and Talman. We show their equivalence by indicating that, generically, both methods generate, up to projection, the same sequence of strategy vectors. In Section 3 we show that the method of van den Elzen and Talman can be used as a practical implementation of the tracing procedure for bi-matrix games. We illustrate this along with an example given in Harsanyi and Selten (1988). We also argue that for nondegenerated bi-matrix games the logarithmic tracing procedure can be mimiced by applying the tracing procedure from a suitably perturbed prior. Finally, we conclude in Section 4 by considering a bi-matrix game in which both players have three actions. We analyze this game with the pivoting procedure. Direct application of the tracing procedure from all possible priors would become very complicated.

2 Comparison of both procedures

In this section we show that for a fixed N-person game, the pivoting method and the linear tracing procedure generically generate identical paths from a given prior to a Nash equilibrium. More precisely, they coincide if no degeneracies occur along the path. However, one type of degeneracy gives no problem for the pivoting procedure, while the linear tracing procedure bifurcates. All degeneracies can be solved for the tracing procedure by adding a logarithmic term to the payoffs.

Let us start with some preliminaries. For given integer $s > 0$, we denote by I_s the set $\{1, \ldots, s\}$. Furthermore, $\mathbb{R}^m_+$ stands for the nonnegative orthant of the m-dimensional Euclidean space, i.e., $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x_r \geq 0, \ \forall r \in I_m\}$. By $e(r)$ we denote the r-th standard unit vector, whose dimension will be clear from the context. Similarly, we denote by $\underline{0}$ and e a vector of zeros and a vector of ones of appropriate length, respectively. Given two vectors x and y in $\mathbb{R}^m$, we denote by $[x, y]$ the line segment of vectors between x and y, i.e., $[x, y] = \{z \in \mathbb{R}^m | z = \lambda x + (1 - \lambda)y, \ 0 \leq \lambda \leq 1\}$. Occasionally, $[x, y]$ denotes a curve connecting x and y, rather than a segment. Also this will be clear from the context. Open and half-open segments are denoted by (x, y) and $(x, y]$ or $[x, y)$, respectively, with obvious meaning. Furthermore, by $x^\top$ ($A^\top$) we mean the transpose of a vector x (matrix A).

Let be given a noncooperative N-person game in normal form. Player i, $i \in I_N$, has n_i pure actions. For $k \in I_{n_i}$, action k of player i is denoted by (i, k). Furthermore, let the set of actions of player i, $i \in I_N$, i.e., $\{(i, 1), \ldots, (i, n_i)\}$, be denoted by $I(i)$. The action set of the game is denoted by $I = \cup_i I(i)$. The total number of actions is denoted by n, i.e., $n = \sum_{i=1}^N n_i$. Thus, the mixed strategy space of player i equals the $(n_i - 1)$-dimensional set $S^{n_i - 1} := \{q_i \in \mathbb{R}^{n_i}_+ | \sum_{k=1}^{n_i} q_{ik} = 1\}$. The strategy space of the game is then $S = \Pi_{i=1}^N S^{n_i - 1}$ with generic element strategy vector $q = (q_1^\top, \ldots, q_N^\top)^\top$. The k-th pure strategy of player i is denoted by e_{ik}. Sometimes, a strategy vector q is denoted as (q_i, q_{-i}), where q_{-i} indicates that player $j \neq i$ plays strategy q_j. Similarly, (q_i, p_{-i}) denotes that player $j \neq i$ plays p_j. If player j, $j \in I_N$, plays pure strategy k_j then player i obtains a payoff $H_i(e_{1k_1}, \ldots, e_{Nk_N})$. From these pure strategy payoffs we obtain for each player i the payoff function $H_i : S \to \mathbb{R}$ defined by

$$H_i(q) = \sum_{\phi \in \Phi} \Pi_{j \in I_N} q_{j\phi_j} H_i(\phi_1, \ldots, \phi_N),$$

where Φ is the set of pure strategy vectors. Thus, the payoff related to a mixed strategy vector q is a weighted average of the relevant pure strategy payoffs. These weights equal the expected probabilities with which the respective pure strategy vectors occur.

In the sequel we denote a noncooperative game in normal form Γ as a tuple $\left(N, \underline{n}, (H_i)_{i=1}^N\right)$, with $\underline{n} = (n_1, \ldots, n_N)$. Now, a strategy vector $q^* \in S$ constitutes a Nash equilibrium of $\Gamma = \left(N, \underline{n}, (H_i)_{i=1}^N\right)$ if

$$H_i(q^*) \geq H_i(q_i, q_{-i}^*), \quad \forall q_i \in S^{n_i - 1} \text{ and } \forall i \in I_N. \tag{2.1}$$

Furthermore, we define the marginal payoff function of player i, $\bar{H}_i : S \to \mathbf{R}^{n_i}$, by

$$\bar{H}_i(q) = (H_i(e_{i1}, q_{-i}), \ldots, H_i(e_{in_i}, q_{-i}))^\top.$$

Thus $\bar{H}_{ik}(q)$, $k \in I_{n_i}$, denotes the payoff to player i if he plays action k whereas player $j \neq i$ plays q_j. Of course, $\forall i \in I_N$ and $\forall q \in S$ it holds that $H_i(q) = q_i^\top \bar{H}_i(q)$. It is well-known that $q^* \in S$ is a Nash equilibrium if and only if

$$q_{ik}^* = 0 \text{ if } \bar{H}_{ik}(q^*) < \max_\ell \bar{H}_{i\ell}(q^*), \quad \forall (i,k) \in I. \tag{2.2}$$

We are now ready to formulate the pivoting process as given in van den Elzen and Talman (1991) for a bi-matrix game. Here we give a straightforward generalization for an N-player game. Starting from a given prior $p \in S$ the procedure generates a piecewise smooth path of strategy vectors q in S satisfying for all $(i,k) \in I$,

$$\begin{array}{lll} q_{ik} = b(q,p)p_{ik} & \text{if} & \bar{H}_{ik}(q) < \max_\ell \bar{H}_{i\ell}(q) \\ q_{ik} \geq b(q,p)p_{ik} & \text{if} & \bar{H}_{ik}(q) = \max_\ell \bar{H}_{i\ell}(q), \end{array} \tag{2.3}$$

where $0 \leq b(q,p) := \min_{(i,\ell)}\{\frac{q_{i\ell}}{p_{i\ell}} | p_{i\ell} > 0\} \leq 1$.

Observe that $q = p$ satisfies (2.3) with $b(p,p) = 1$. Also each Nash equilibrium $q^* \in S$ satisfies (2.3) with $b(q^*, p) = 0$ or with $p_{ik} = 0$ for all (i,k) for which $\bar{H}_{ik}(q^*) < \max_\ell \bar{H}_{i\ell}(q^*)$. Along the path the parameter $b(q,p)$ initially decreases from 1. Let us define for prior p and $b \in [0,1]$, the set $S_p(b) = \{q \in S | q \geq bp\}$. Note that $S_p(1) = \{p\}$ and $S_p(0) = S$. Then we derive from (2.1), (2.2) and (2.3) that the pivoting procedure generates from p a path of restricted Nash equilibria, i.e., Nash equilibria with strategies restricted to the set $S_p(b)$ with b going from 1 either to zero or to some $\bar{b} > 0$. The vectors generated by the pivoting procedure are Nash equilibria in the sense that nonoptimal actions are played with the minimally allowed probability. In case the procedure stops at $\bar{b} > 0$, the point q^* related to $S_p(\bar{b})$ is such that $p_{ik} = 0$ for all (i,k) for which $\bar{H}_{ik}(q^*) < \max_\ell \bar{H}_{i\ell}(q^*)$, and thus q^* is a Nash equilibrium of the game.

For later reference we rewrite (2.3) as follows. Firstly, we define $\forall i \in I_N$ and $\forall q \in S$, the set $T_i(q)$ collecting the optimal actions of player i at strategy vector q, i.e., $T_i(q) = \{(i,k) \in I(i) | \bar{H}_{ik}(q) = \max_\ell \bar{H}_{i\ell}(q)\}$. We then derive that the set of strategy vectors satisfying (2.3) equals the set of strategy vectors $q \in S$ satisfying $\forall i \in I_N$,

$$q_i = b(q,p)p_i + \sum_{(i,k)\in T_i(q)} \lambda_{ik} e(k), \tag{2.4}$$

where $\lambda_{ik} \geq 0$ and $\sum_{(i,k)\in T_i(q)} \lambda_{ik} = 1 - b(q,p)$.

The procedure of van den Elzen and Talman (1991) is well-defined for almost all priors, given an arbitrary game. In those cases system (2.3) determines a piecewise smooth path connecting the prior and a Nash equilibrium. This path is piecewise linear in case of two players. Problems might arise when degeneracies occur, i.e., when along the path traced by the procedure more than one inequality in (2.3) becomes an equality simultaneously. In case of two players these problems can be resolved by standard techniques used in linear programming.

Next, we arrive at the linear tracing procedure, which also starts with a prior p. Here p_i is interpreted as the expectations of players $j \neq i$ about the strategy played by i. The linear tracing procedure generates a path of Nash equilibria for games Γ_p^t from $t = 0$ to $t = 1$. Here Γ_p^1 denotes the original game $\Gamma = \left(N, \underline{n}, (H_i)_{i=1}^N\right)$. The payoff function of player i in game Γ_p^t, $0 \leq t \leq 1$, is given by $H_i^t : S \mapsto \mathbf{R}$, with

$$H_i^t(q;p) = tH_i(q) + (1-t)H_i(q_i, p_{-i}). \tag{2.5}$$

At $t = 0$ each player i plays optimally against p_{-i}. For the linear tracing procedure to be well-defined this optimal reply has to be unique. This generically holds. It follows directly from (2.5) that if q^t is a Nash equilibrium point of Γ_p^t, then for each $i \in I_N$, q_i^t has to be a best reply against the mixed strategy $tq_{-i}^t + (1-t)p_{-i}$. The linear tracing procedure is well-defined in almost all cases in which the pivoting procedure is well-defined.

We return to this point in the next theorem.

Summarizing the foregoing we can say that the pivoting procedure generates a path $B(\Gamma, p)$ of vectors in $\mathcal{B}(\Gamma, p)$, where

$$\mathcal{B}(\Gamma, p) = \{q \in S | q \in NE_{S_p(b)}(\Gamma),\ b \in [0, 1]\},$$

with $NE_{S_p(b)}(\Gamma)$ the set of Nash equilibria of Γ with restricted strategy set $S_p(b)$. More precisely, the path connects $S_p(1) = \{p\}$ and a Nash equilibrium of the original game.

On the other hand, the linear tracing procedure generates a path of vectors $L(\Gamma, p)$ in $\mathcal{L}(\Gamma, p)$, where

$$\mathcal{L}(\Gamma, p) = \{(q^t, t) \in S \times [0, 1] | q^t \in NE(\Gamma_p^t)\},$$

with $NE(\Gamma_p^t)$ the set of Nash equilibria of Γ_p^t. The path connects $S \times \{0\}$ and $S \times \{1\}$. The relation between the two expressions is revealed by the next theorem.

Theorem 2.1. Let be given an N-person noncooperative game Γ and a prior $p \in S$. If the linear tracing procedure is well-defined, then the paths of the pivoting procedure and the linear tracing procedure coincide up to projection. More precisely, a Nash equilibrium of Γ_p^t, $0 \leq t \leq 1$, generated by the linear tracing procedure, is in 1-1 correspondence with a restricted Nash equilibrium on $S_p(1-t)$ generated by the pivoting procedure. Furthermore, one source for bifurcation of the linear tracing procedure is removed by the pivoting procedure.

Proof. The linear tracing procedure generates $\forall t \in [0, 1]$, vectors $q^t \in NE(\Gamma_p^t)$. The latter implies that $\forall i \in I_N$, q_i^t is optimal against $tq_{-i}^t + (1-t)p_{-i}$, i.e., $(1-t)p + tq^t \in NE_{S_p(1-t)}(\Gamma)$. The latter vector satisfies (2.4) with $b(q, p) = 1 - t$.

On the other hand, the pivoting procedure generates vectors $q \in NE_{S_p(b)}(\Gamma)$ with $b \in [0, 1]$, i.e., $q_i = bp_i + \sum_{T_i(q)} \lambda_{ik} e(k)$, with $\sum_{T_i(q)} \lambda_{ik} = 1 - b$, $\forall i \in I_N$. Therefore, $q = bp + (1-b)\bar{q}$, with $\bar{q}_i = \frac{1}{1-b} \sum_{T_i(q)} \lambda_{ik} e(k)$, $\forall i \in I_N$, if $b \neq 1$. Thus, $q \in NE(\Gamma_p^{1-b})$. In case $b = 1$, $\bar{q}_i = e(k_i)$, $\forall i \in I_N$, with k_i the optimal action of player i against p.

Clearly, the relations above describe a homeomorphism between Nash equilibria of Γ_p^t and restricted Nash equilibria of Γ on $S_p(1-t)$.

Finally, observe that the linear tracing procedure traces a path from $t = 0$ to $t = 1$. This corresponds to the variable $b(q, p)$ in the pivoting procedure going from 1 to 0.

However, the latter procedure may stop at a q^* at which $b(q^*, p) > 0$. This happens if $p_{ik} = 0$, $\forall (i,k) \notin \cup_j T_j(q^*)$. In that case the tracing procedure may bifurcate. If not, the path $L(\Gamma, p)$ becomes nonlinear because q^t is nonlinear in t for $t > 1 - b(q^*, p)$. □

In case the linear tracing procedure bifurcates, Harsanyi and Selten propose to replace that method by the logarithmic tracing procedure. In the latter procedure the payoffs are augmented with a logarithmic term. More precisely, $\forall t \in [0,1]$, we define the game $\bar{\Gamma}^t_{p\eta}$ with payoff function $\bar{H}^t_i : S \mapsto \mathbf{R}$, $\forall i \in I_N$, given by

$$H^t_i(q; p, \eta) = tH_i(q) + (1-t)H_i(q_i, p_{-i}) + \eta(1-t)\alpha_i \sum_{k=1}^{n_i} \log q_{ik}, \qquad (2.6)$$

where η is a very small positive number and α_i a game-dependent positive constant. It can be shown that for any game Γ, any prior p and any small positive number η there exists a unique path of Nash equilibria $\bar{q}(t, p, \eta)$ in $\Gamma^t_{p\eta}$ with t going from 0 to 1. Let us denote the end point by $\bar{q}(1, p, \eta)$. Then the limit of $q(1, p, \eta)$ with η going to zero, exists and is the equilibrium selected by the logarithmic tracing procedure. Furthermore, it equals the equilibrium selected by the linear tracing procedure in case the latter procedure is well-defined. For the proof of all of this we refer to Harsanyi and Selten (1988) and Schanuel et al. (1991). We stress that in case the pivoting procedure is well-defined while the linear tracing procedure bifurcates, the equilibrium selected by our procedure may differ from the one obtained by the logarithmic tracing procedure. This will be illustrated in the next section.

One of the goals of the Harsanyi-Selten selection method is to find a perfect Nash equilibrium. That is why they apply the method to a sequence of truncated games, i.e., games in which actions are always played with positive probability. Then by letting the disturbances go to zero, the corresponding sequence of Nash equilibria gives a perfect Nash equilibrium. All this would not be necessary in case the logarithmic tracing procedure would always give a perfect Nash equilibrium, which is not the case (see van Damme (1983, p. 85)). However, the pivoting procedure always yields a perfect equilibrium whenever the prior is completely mixed. In van den Elzen and Talman (1991) this is proved for bi-matrix games. But the proof is in fact also valid for any finite number of players. From Theorem 2.1 we obtain that this result also applies to the linear tracing procedure. This result is the more relevant when we consider the tracing procedure of Harsanyi and Selten (1988) in the light of their complete selection theory. One part of that theory is devoted to the choice of the prior. This prior will always be completely mixed. This is due to the fact that we are dealing with (sub)games in which there are no dominated actions. Thus, initially there will be put some weight on any action.

For games with more than two players all methods are computationally cumbersome. This because all equations become nonlinear. However, in van den Elzen and Talman (1994) a method is described that makes use of linearizations. That method can be used for approximating a Nash equilibrium of a noncooperative N-person game. However, the appealing interpretation is lost.

3 Application to bi-matrix games

For bi-matrix games the marginal payoff functions of the players are linear. We will show that in this case the procedure of van den Elzen and Talman (1991) boils down to a complementary pivoting procedure in a system of linear equations. This procedure can easily be implemented on a computer. We already argued in the previous section that the procedure generically coincides, up to projection, with the linear tracing procedure. In that respect our method can also be viewed as a computationally attractive method for implementing the linear tracing procedure for bi-matrix games. However, in Section 2 we indicated one important difference between the pivoting method and the linear tracing procedure. The latter method always traces a path from $t = 0$ to $t = 1$. The pivoting method may stop earlier. This happens if at an intermediate stage a Nash equilibrium for the original game has been reached. The path generated by the linear tracing procedure from the vector related to that Nash equilibrium onwards, bifurcates in case of more equilibria. In addition, it always becomes nonlinear and is therefore hard to follow computationally. Of course, also the logarithmic tracing procedure provides a solution in these cases. But that method seems to be of no practical use in computation. This because we have to follow nonlinear paths.

A bi-matrix game Γ is a tuple (n_1, n_2, U, V) with n_i the number of actions of player i, $i = 1, 2$, whereas U (V) denotes the payoff matrix of player 1 (2). More precisely, U and V are of dimension $n_1 \times n_2$ where an element u_{jk} of U (v_{jk} of V) indicates the payoff to player 1 (2) if player 1 plays action $j \in I_{n_1}$ and player 2 plays action $k \in I_{n_2}$. For the marginal payoff function $\bar{H} : S \rightarrow \mathbb{R}^{n_1+n_2}$, we obtain

$$\bar{H}(q) = (\bar{H}_1^\top(q), \bar{H}_2^\top(q))^\top, \text{ with } \bar{H}_1(q) = Uq_2 \text{ and } \bar{H}_2(q) = V^\top q_1. \tag{3.1}$$

Now, consider (2.3) and observe that for each q in S generated by the pivoting method starting from prior $p \in S$, there is at least one set $T \subset I$ satisfying

$$\begin{array}{lll} q_{ik} \geq bp_{ik} & \text{and} & \bar{H}_{ik}(q) = \beta_i, \quad (i,k) \in T \\ q_{ik} = bp_{ik} & \text{and} & \bar{H}_{ik}(q) \leq \beta_i, \quad (i,k) \notin T, \end{array} \tag{3.2}$$

with $T_i := T \cap I(i) \neq \emptyset$, $i = 1, 2$. In the above b is a variable substituting $b(q, p)$, whereas $\beta_1 = \max_k U^k q_2$ and $\beta_2 = \max_k q_1^\top V_k$ with C^k (C_k) denoting the k-th row (column) of a matrix C.

Observe from (3.1) that $\bar{H}_{1k}(q) = U^k q_2$ and $\bar{H}_{2k}(q) = q_1^\top V_k$. Next, we substitute (3.1) into (3.2), add slacks $\mu_{ik} \geq 0$ for the inequalities $\bar{H}_{ik}(q) \leq \beta_i$, $(i,k) \notin T$, and slacks $\lambda_{ik} \geq 0$ for the inequalities $q_{ik} \geq bp_{ik}$, $(i,k) \in T$. We obtain that each q on the path satisfies $\forall i \in I_2$

$$q_i = bp_i + \sum_{(i,k) \in T_i} \lambda_{ik} e(k) \text{ with } \sum_{(i,k) \in T_i} \lambda_{ik} = 1 - b, \tag{3.3}$$

while

$$\begin{array}{lllll} Uq_2 & + & \sum_{(1,k) \notin T} \mu_{1k} e(k) & = & e\beta_1 \\ V^\top q_1 & + & \sum_{(2,k) \notin T} \mu_{2k} e(k) & = & e\beta_2. \end{array} \tag{3.4}$$

By substituting (3.3) in (3.4) we obtain the system

$$b \begin{bmatrix} Up_2 \\ V^\top p_1 \\ 1 \\ 1 \end{bmatrix} + \sum_{(1,k) \in T} \lambda_{1k} \begin{bmatrix} \underline{0} \\ V_k^\top \\ 1 \\ 0 \end{bmatrix} + \sum_{(2,k) \in T} \lambda_{2k} \begin{bmatrix} U_k \\ \underline{0} \\ 0 \\ 1 \end{bmatrix} + \sum_{(1,k) \notin T} \mu_{1k} \begin{bmatrix} e(k) \\ \underline{0} \\ 0 \\ 0 \end{bmatrix}$$

$$+ \sum_{(2,k) \notin T} \mu_{2k} \begin{bmatrix} \underline{0} \\ e(k) \\ 0 \\ 0 \end{bmatrix} - \beta_1 \begin{bmatrix} e \\ \underline{0} \\ 0 \\ 0 \end{bmatrix} - \beta_2 \begin{bmatrix} \underline{0} \\ e \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ 1 \\ 1 \end{bmatrix}, \tag{3.5}$$

where $0 \leq b \leq 1$, $\lambda_{ik} \geq 0$, $\forall (i,k) \in T$, and $\mu_{ik} \geq 0$, $\forall (i,k) \notin T$. Observe that system (3.5) contains $n_1 + n_2 + 2$ equations with $n_1 + n_2 + 3$ variables, thus leaving one degree of freedom. Clearly, any strategy vector q satisfying (2.3) fits into (3.5) for adequate values of the coefficients. The pivoting procedure generates a piecewise linear path of strategy vectors by performing a sequence of linear programming pivoting steps in system (3.5). Under the standard nondegeneracy assumption it always converges to a Nash equilibrium. Otherwise, we can resolve degeneracy problems by applying standard techniques.

Let us quickly recall the working of this method. At the start we have $b = 1$, $T = \emptyset$, $\beta_1 = U^{k_1} p_2 = \max_k U^k p_2$, $\beta_2 = p_1^\top V_{k_2} = \max_k p_1^\top V_k$, $\mu_{1k} = \beta_1 - U^k p_2$ for $k \neq k_1$ and $\mu_{2k} = \beta_2 - p_1^\top V_k$ for $k \neq k_2$. From (3.3) we deduce that indeed $q = p$. The procedure then leaves p by decreasing b from 1 and therefore increasing λ_{1k_1} and λ_{2k_2}. If in system (3.5) the variable b becomes zero then the procedure has found a Nash equilibrium (see discussion below (2.3)). Else, if in (3.5) a μ-variable becomes zero - say μ_{ih} - then either $p_{jk} = 0$ for all (j,k) with $\mu_{jk} \neq 0$ and a Nash equilibrium has been found (see below (2.3)) or in the next iteration λ_{ih} is increased from zero. Conversely, the variable μ_{ik} is increased from zero in the next iteration if λ_{ik} becomes zero.

We can give some game-theoretic interpretation for the procedure. From (3.2) we see that T constitutes the set of actions giving maximal marginal payoff to a player. For convenience we call these actions optimal. At the start each player has a unique optimal action in case the problem is nondegenerated. From the start the related probabilities are increased (λ_{1k_1} and λ_{2k_2} are raised from zero), whereas the probabilities related to the other actions are decreased relatively equally (decrease in b). If an action (j,h) becomes optimal (μ_{jh} becomes zero) then a Nash equilibrium may be reached. Otherwise, it is kept optimal and the related probability is relatively increased (λ_{jh} is raised from zero). On the other hand, if the probability with which an optimal action (i,k) is played becomes relatively minimal (λ_{ik} becomes zero) then it is kept so and the related action is made nonoptimal (μ_{ik} is increased from zero).

Let us now review both procedures along with the example given by Harsanyi and Selten (1988). They consider the bi-matrix game $\Gamma = (n_1, n_2, U, V)$ with $n_1 = n_2 = 2$ and

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

It is easily checked that this game has two pure Nash equilibria, $A = ((1,0)^\top, (1,0)^\top)$ and $C = ((0,1)^\top, (0,1)^\top)$, and one mixed equilibrium, $Q = ((\frac{4}{5}, \frac{1}{5})^\top, (\frac{1}{3}, \frac{2}{3})^\top)$. In Figure 3.1 we represent all relevant information in the strategy space. The pure strategy vectors $((0,1)^\top, (1,0)^\top)$ and $((1,0)^\top, (0,1)^\top)$ are denoted by B and D, respectively. The index (k_1, k_2) denotes that in the related subset of S action k_1 is optimal for player 1 and action k_2 for player 2. For example, (1,2) holds at strategy vectors in the convex hull of $\{B, F, Q, J\}$. Similarly, we obtain that on $[J, Q]$ action 2 is optimal for player 2 whereas player 1 is indifferent between his actions. Harsanyi and Selten (1988) consider three different priors, $\hat{p} = ((\frac{1}{3}, \frac{2}{3})^{\top}, (\frac{1}{6}, \frac{5}{6})^{\top})$, $p' = ((\frac{1}{2}, \frac{1}{2})^{\top}, (\frac{2}{3}, \frac{1}{3})^{\top})$, and $p'' = ((\frac{3}{5}, \frac{2}{5})^{\top}, (\frac{2}{3}, \frac{1}{3})^{\top})$. These vectors are also indicated in the Figure 3.1. Furthermore, we depict the set $S_{p'}(\frac{3}{4})$. Going from $b = 1$ to $b = 0$, the set $S_{p'}(b)$ is expanding from $\{p'\}$ to S.

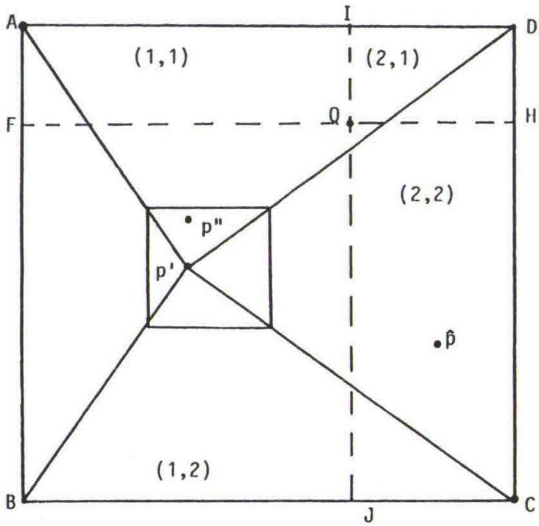

Figure 3.1. The strategy space S of the game with the three equilibria A, C, and Q.

In the Figures 3.2.a and 3.2.b the case with prior equal to $\hat{p}$ is treated. Figure 3.2.a deals with the procedure of van den Elzen and Talman (1991). The set of strategy vectors satisfying (3.2) consists of two disjoint paths.

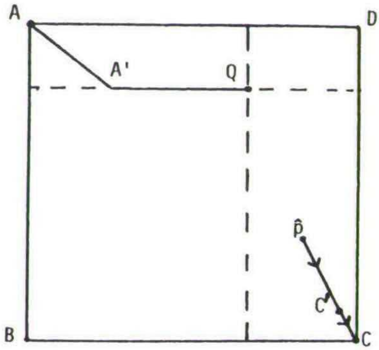

Figure 3.2.a. The restricted Nash equilibria of Γ related to the sets $S_{\hat{p}}(b), b \in [0,1]$, consist of the segments $[\hat{p}, C], [A, A']$, and $[A', Q]$.

One path connects the prior $\hat{p}$ and C, the second piecewise linear path connects A and Q. It is easily verified that the vectors on these paths indeed satisfy (3.2). For example, on $[A', Q]$ the set T equals $\{(1,1),(2,1),(2,2)\}$. Of course, the procedure generates the path from $\hat{p}$ to the selected equilibrium C. Figure 3.2.b is taken from Harsanyi and Selten (1988) and reveals the set of Nash equilibria of $\Gamma^t_{\hat{p}}$, $t \in [0,1]$, in $S \times [0,1]$, containing the path traced by the linear tracing procedure. The latter path connects $C \times \{0\}$ and $C \times \{1\}$. The other part corresponds to Nash equilibria for $\Gamma^t_{\hat{p}}$, $t \geq \frac{7}{10}$. Note that the curve connecting R and Q is nonlinear.

Remark that our procedure has also advantages concerning its graphical representation. For the method of Harsanyi and Selten we need one extra dimension to represent

t. In our method that variable is reflected in the expanding set around the prior.

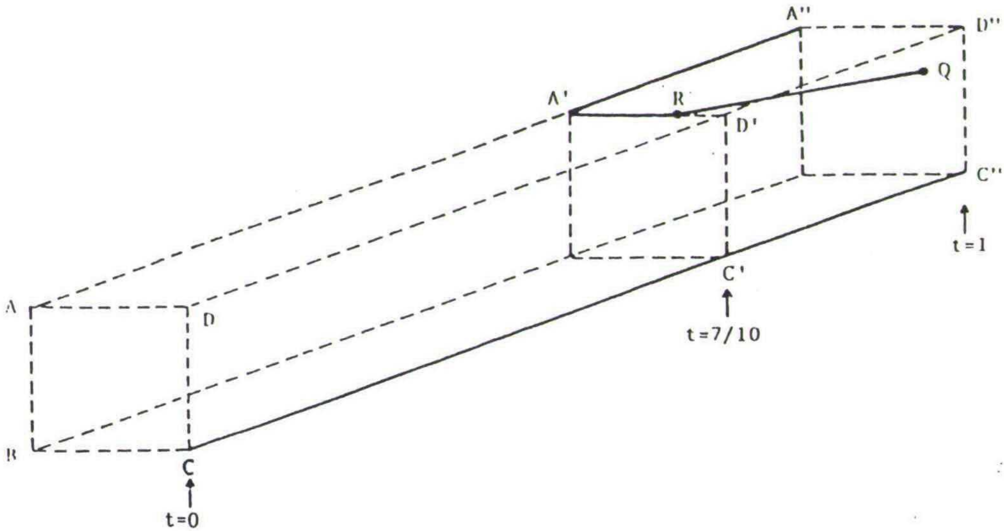

Figure 3.2.b. The set of Nash equilibria of $\Gamma_{\hat{p}}^{t}$, $t \in [0,1]$, depicted in $S \times [0,1]$.

As stipulated in Section 2 every q on the paths in Figure 3.2.a can be seen as a restricted Nash equilibrium point of $S_{\hat{p}}(1-t)$, for some $t \in [0,1]$. The strategy vectors on the piecewise linear path connecting Q and A are restricted equilibria on $S_{\hat{p}}(1-t)$, $t \geq \frac{7}{10}$, and correspond to Nash equilibria of $\Gamma_{\hat{p}}^{t}$, $t \geq \frac{7}{10}$, as depicted in Figure 3.2.b. Observe that in our set up the latter path is much easier to trace than the related path of the linear tracing procedure, where nonlinearities in t originating from expressing Q as an equilibrium of $\Gamma_{\hat{p}}^{t}$, $t > \frac{7}{10}$, are involved.

Similarly, we can analyze the case in which the prior equals p'. The corresponding graphical representations for the pivoting procedure and the linear tracing procedure are given in Figures 3.3.a and 3.3.b, respectively. Now, the piecewise linear curve with segments $[p', D']$, $[D', C'']$, and $[C'', C]$ in Figure 3.3.a corresponds to the piecewise linear curve connecting D, D', C'', and C''' in Figure 3.3.b. Similarly, the piecewise linear curve with segments $[A, A']$ and $[A', Q]$ in Figure 3.3.a relates to the curve consisting of the segments $[Q, R]$, $[R, A'']$, and $[A'', A''']$ in Figure 3.3.b.

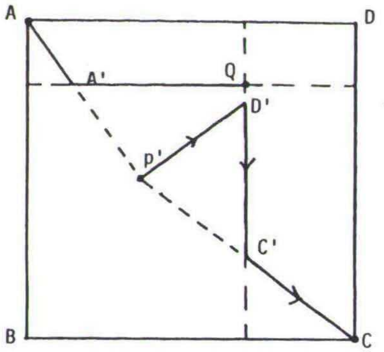

Figure 3.3.a. The set of restricted Nash equilibria of Γ on $S_{p'}(b)$ for $b \in [0,1]$. The piecewise linear path connecting p', D', C', and C is traced by the pivoting procedure.

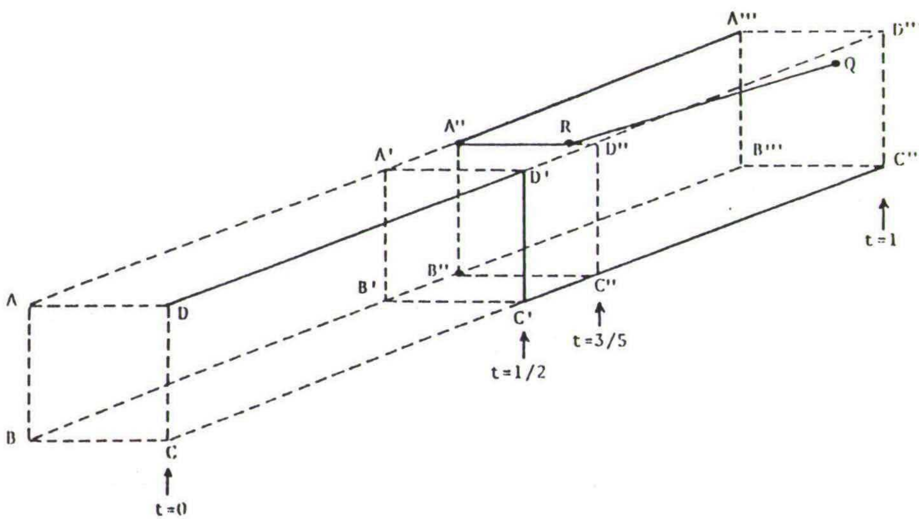

Figure 3.3.b. The set of Nash equilibria of $\Gamma^t_{p'}$ for $t \in [0,1]$ represented in $S \times [0,1]$. The path connecting D, D', C', and C''' is followed by the linear tracing procedure.

Let us finally move to the case with prior p''. The set of strategy vectors satisfying the conditions of the pivoting procedure are depicted in Figure 3.4.a. This procedure yields within one iteration the mixed Nash equilibrium Q. That the set of vectors satisfying (2.3) bifurcates at Q is not relevant for this method because the procedure stops at Q with $b(Q, p'') = \frac{1}{2} > 0$. Such a b corresponds to $t = \frac{1}{2}$ in the linear tracing procedure. But for $t = \frac{1}{2}$ that procedure generates two segments of Nash equilibria corresponding to $[A', Q]$ and $[Q, C']$ and it therefore bifurcates. This is depicted for the linear tracing procedure in Figure 3.4.b with two curves, one connecting D', C'', and C''', the other connecting Q, D', A', and A''. In addition, the linear tracing procedure involves nonlinearities, in casu the curve $[D', Q]$.

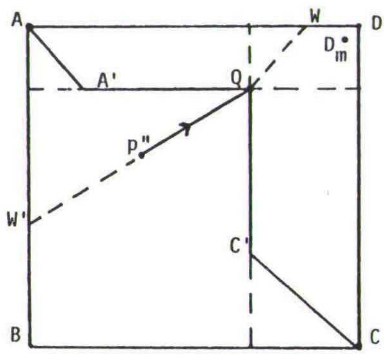

Figure 3.4.a. The set of restricted Nash equilibria of Γ on $S_{p''}(\frac{1}{2})$ consists of the segments $[A', Q]$ and $[Q, C']$.

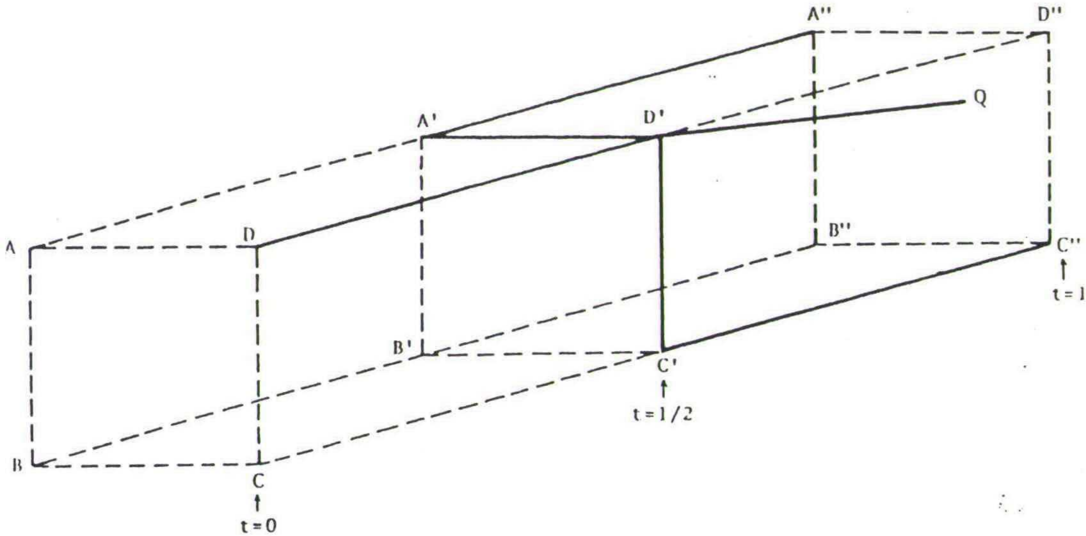

Figure 3.4.b. The set $\mathcal{L}(\Gamma, p'')$ revealing a bifurcation at $t = \frac{1}{2}$.

To resolve the bifurcation problem Harsanyi and Selten (1988) developed the logarithmic tracing procedure which is well-defined for any game and any prior, i.e., it always selects a unique equilibrium. However, that method is not very tractable. We argue that for generic bi-matrix games, a specific perturbation of the prior could also do the job. More precisely, if in case of bifurcation the linear tracing procedure would be restarted from a specifically perturbed prior, it selects the same equilibrium as the logarithmic tracing procedure starting from the original prior. Of course, one can also apply our method directly using standard techniques to avoid degeneracy, but this may lead to another Nash equilibrium.

Consider expression (2.6) concerning the logarithmic payoffs. From the appearance of the logarithmic term it is clear that the related Nash equilibria for $t < 1$ are completely mixed. Compared to the Nash equilibria related to the payoff structure in (2.5), the latter equilibria are perturbed to the centre of the strategy space. In terms of restricted Nash equilibria, each equilibrium generated from (2.6) for given p and η again corresponds to a restricted Nash equilibrium for that payoff structure on $S_p(1-t)$. In comparison to the pivoting procedure also that vector is slightly perturbed towards the centre of $S_p(1-t)$. In the sequel we only consider standard nondegenerated bi-matrix games with a finite number of isolated equilibria. These games constitute a dense subset of the set of bi-matrix games. Concerning these games it holds that there exists a finite set of (relatively) open subsets in the strategy set, such that starting the linear tracing procedure from any prior within a certain subset leads to one specific selected Nash equilibrium. The closures of these open sets of attraction cover the strategy set. Degeneracies occur at intersections of these closures. All this is derived from the equivalence of the linear tracing procedure and the pivoting procedure in case the first procedure is well-defined. Because in case of bi-matrix games the pivoting procedure operates in a linear system the observations given above follow from standard linear programming theory. In the theorem below we state that if we apply the logarithmic tracing procedure from a degenerated prior whereas the path of restricted Nash equilibria related to (2.6) for small η enters a specific attraction set, we could alternatively apply the linear tracing procedure from a prior within that set.

Theorem 3.1. Let be given a nondegenerated bi-matrix game Γ and prior p, and let the linear tracing procedure be not well-defined for (Γ, p). For given t and η, let $\hat{q}(t,p,\eta)$ be the restricted Nash equilibrium on $S_p(1-t)$ related to the logarithmic tracing procedure. Furthermore, assume the existence of some small $\tilde{t}$ and some small $\tilde{\eta}$ such that all $\hat{q}(t,p,\eta)$, $0<t<\tilde{t}$ and $0<\eta<\tilde{\eta}$ lie in one attraction set. Then the logarithmic tracing procedure with prior p selects the same Nash equilibrium as the linear tracing procedure with a prior within that attraction set.

Proof. For any small $\eta>0$ there exists a curve $\hat{q}(t,p,\eta)$ of restricted Nash equilibria on $S_p(1-t)$ related to Nash equilibria for payoffs $H_i^t(q;p,\eta)$, $i \in I_2$, with $t \in [0,1]$. Now, fix a small $\bar{t} \in (0,\tilde{t}]$ and $\bar{\eta} \in (0,\tilde{\eta}]$ and consider the vector $\hat{q}(\bar{t},p,\eta)$. For $\bar{t}$, η small enough it holds that $\hat{q}(1,p,\bar{\eta}) = \lim_{t\to 1}\hat{q}(t,p,\bar{\eta}) = \lim_{t\to 1}\hat{q}(t,\hat{q}(\bar{t},p,\eta),\eta)$. Thus, for small $\bar{\eta}$ the path from p has the same limit as the path starting from $\hat{q}(\bar{t},p,\bar{\eta})$, i.e., a vector early on the former path. This is due to the fact that $S_p(1-t) \approx S_{\hat{q}(\bar{t},p,\bar{\eta})}(1-t)$ and $H_i^t(q;\hat{q}(\bar{t},p,\bar{\eta}),\bar{\eta}) = H_i^t(q;p,\bar{\eta}) + (1-t)\,(H_i(q;\hat{q}(\bar{t},p,\bar{\eta})) - H_i(q;p))$. The influence of the latter term disappears if $t\to 1$. Observe that this rewriting is only possible because for bi-matrix games, H_i is linear in q_i.

Repeating this for $\bar{\eta} \downarrow 0$ we obtain that $\hat{q}(1,p) = \lim_{\bar{\eta}\downarrow 0}\hat{q}(1,p,\bar{\eta}) = \lim_{\bar{\eta}\downarrow 0}\hat{q}(1,\hat{q}(\bar{t},p,\bar{\eta}),\bar{\eta})$. Here $\hat{q}(1,p)$ is the equilibrium selected by the logarithmic tracing procedure from p. Because for small enough t and η, all $\hat{q}(t,p,\eta)$ lie in the same attraction set, $\lim_{\eta\downarrow 0}\hat{q}(1,\hat{q}(t,p,\eta),\eta) = \lim_{\eta\downarrow 0}\hat{q}(1,\hat{q}(\tilde{t},p,\tilde{\eta}),\eta)$, for small enough $\tilde{t}$ and $\tilde{\eta}$. □

We illustrate this for the case sketched in Figure 3.4.a. Applying the linear tracing procedure from p'' gives a projected movement towards D, i.e., for player 1 action 1 is optimal, and for player 2 action 2. However, the logarithmic tracing procedure gives an optimal strategy vector equal to D_m being completely mixed and lying on the diagonal of S, connecting B and D. Thus, initially the logarithmic tracing procedure moves - when projected - towards D_m. But then it enters the attraction set related to C and it selects C. Note that the attraction set for the logarithmic tracing procedure related to C consists of all strategy vectors below and on $\{[W',Q],[Q,W]\}$. For the linear tracing procedure we obtain the same set except $[W',Q]$ and $[Q,W]$. Thus, if we move p'' somewhat into the direction of D_m and apply the linear tracing procedure then we select the same equilibrium. If we directly apply the pivoting procedure starting from a prior on $\{[W',Q],[Q,W]\}$ always Q is selected. Hence, for those priors, the equilibrium selected by the pivoting procedure differs from that selected by the logarithmic tracing procedure.

Finally we remark that Theorem 3.1 will not always be applicable. If the payoff structure of the game is such that in Figure 3.4.a. the piecewise linear curve consisting of the segments $[W',Q]$ and $[Q,W]$ coincides with the diagonal $[B,D]$ then no attraction set will be entered. The paths related to the logarithmic tracing procedure remain on the diagonal and the mixed equilibrium Q will be selected. This case is considered by Harsanyi and Selten (1988, Section 4.14).

4 An example

The main importance of our constructive approach is that it enables us to use the tracing procedure in practice beyond (2×2) bi-matrix games. We illustrate this by analyzing the bi-matrix game introduced by Young (1993), in which each player has three actions. The payoff matrices are given by

$$U = \begin{bmatrix} 6 & 5 & 0 \\ 0 & 7 & 5 \\ 0 & 5 & 8 \end{bmatrix} \text{ and } V = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 7 & 5 \\ 0 & 5 & 8 \end{bmatrix}.$$

This game has three pure Nash equilibria, $((1,0,0)^\top,(1,0,0)^\top)$, $((0,1,0)^\top,(0,1,0)^\top)$, and $((0,0,1)^{\top},(0,0,1)^{\top})$. The other two equilibria are $((\frac{7}{8},\frac{1}{8},0)^{\top},(\frac{7}{8},\frac{1}{8},0)^{\top})$ and $((0,\frac{3}{5},\frac{2}{5})^{\top},(0,\frac{3}{5},\frac{2}{5})^{\top})$.

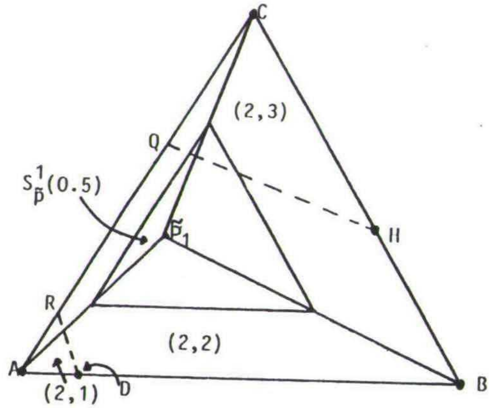

Figure 4.1.a. The strategy space of player 1.

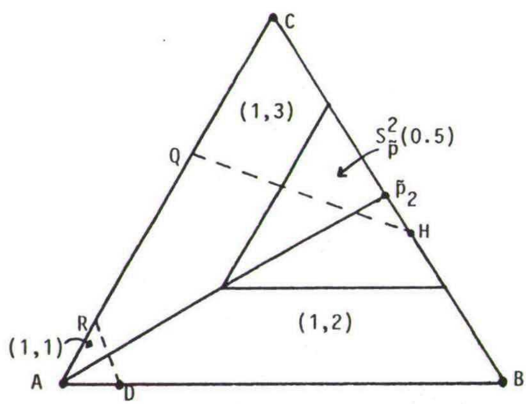

Figure 4.1.b. The strategy space of player 2.

In the Figures 4.1.a and 4.1.b the strategy spaces of players 1 and 2, respectively, are depicted. The indices (i,k) denote optimal reply actions for player i. For example, the index $(2,3)$ in Figure 4.1.a indicates that action 3 is the best reply of player 2 against any strategy in the convex hull of C, Q, and H played by player 1. Similarly, $(1,1)$ in Figure 4.1.b denotes that action 1 is optimal for player 1 against any strategy of player 2 in the convex hull of A, R, and D. The mixed equilibria are indicated by (D,D) and (H,H). Furthermore, $S_{\bar{p}}(0.5)$ equals $S_{\bar{p}}^2(0.5) \times S_{\bar{p}}^2(0.5)$, with the latter subsets as indicated in the figures.

Let us now consider how the linear tracing procedure operates on this game. We do this by applying the pivoting method. For some priors $p = (p_1, p_2)$, the analysis goes very easy. For example, if p_1 lies in the area corresponding to $(2,3)$ and p_2 in the region related to $(1,3)$, then the Nash equilibrium (C,C) is reached in one step. Similarly, we obtain that (B,B) is reached in one step from a prior p with p_1 related to $(2,2)$ and p_2 related to $(1,2)$. Finally, with p_1 corresponding to $(2,1)$ and p_2 to $(1,1)$, the equilibrium (A,A) is obtained. To get further insight we consider two more priors, $\hat{p} = ((\frac{1}{2},\frac{1}{8},\frac{3}{8})^\top,(\frac{1}{8},\frac{1}{8},\frac{3}{4})^\top)$ and $\tilde{p} = ((\frac{1}{2},\frac{1}{8},\frac{3}{8})^\top,(0,\frac{1}{2},\frac{1}{2})^\top)$. Both priors are indicated in the Figures 4.2.a and 4.2.b.

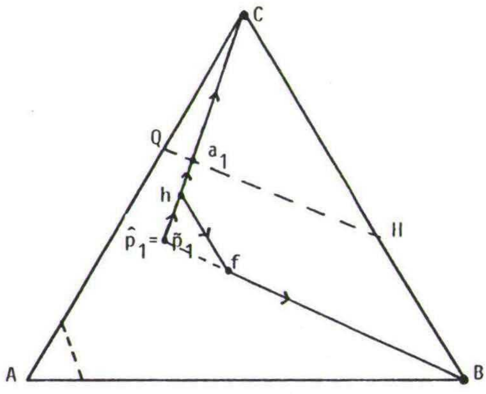

Figure 4.2.a. The strategy space of player 1.

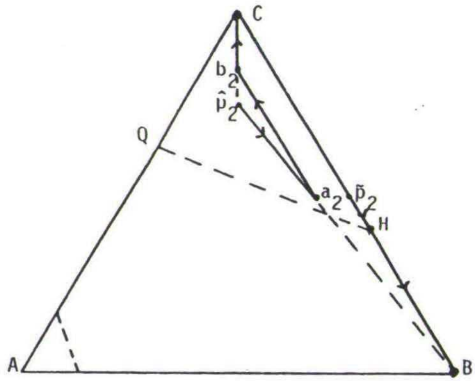

Figure 4.2.b. The strategy space of player 2.

Given prior $\hat{p}$ it is optimal for player 2 to play (2,2) and for player 1 to play (1,3). Thus, in the pivoting procedure the related probabilities are increased. Along the strategy path from $\hat{p}$ the component related to player 1 moves towards C and that of player 2 towards B. At $t = \frac{13}{37}$ the strategy of player 1 equals the point a_1, that of player 2 equals the point a_2, while also the third action of player 2 becomes optimal. Next, the related probability is increased and the strategy of player 2 moves towards b_2 while player 1 stays at a_1 and t remains fixed. In terms of restricted Nash equilibria we derive that $NE_{S_{\hat{p}}(\frac{13}{27})}(\Gamma)$ equals $\{q \in S | q_1 = a_1,\ q_2 \in [a_2, b_2]\}$. When the strategy of player 2 becomes equal to b_2, the procedure continues by making his second action nonoptimal. This simultaneously means that the path related to player 1 moves towards C. In this way the equilibrium (C, C) is obtained when starting from $\hat{p}$.

Analogous reasonings can be made when starting with prior $\tilde{p}$. From $\tilde{p}$ the path related to player 1 moves towards C, whereas the strategy of player 2 moves into the direction of B. At $t = \frac{1}{5}$, the strategy of player 2 becomes equal to H and equal to h for player 1. Then, for player 1 also his second action becomes optimal, the related probability is increased and his path of strategies moves towards f. Meanwhile, t remains constant and the strategy of player 2 stays at H. At f the third action of player 1 is made nonoptimal and the strategies of both players move towards B. In this manner the equilibrium (B, B) is reached.

Evaluating the above we may conclude the following. In case the prior p is such that

p_1 lies in the region with label (2,2) and p_2 in the region with label (1,3), it is especially relevant whether action (2,3) becomes optimal first or action (1,2). In the former case the equilibrium (C, C) is selected, in the latter case (B, B). If both (2,3) and (1,2) become optimal at the same time then equilibrium (H, H) is reached.

Similar remarks hold for instances with priors p such that p_1 is in region (2,3), (2,2) or (2,1), whereas p_2 lies in (1,2), (1,1), and (1,2), respectively. In case the prior p is such that either p_1 lies in region (2,1) and p_2 in region (1,3), or p_1 in region (2,3) and p_2 in region (1,1), then always the Nash equilibrium (B, B) is selected. This also occurs in case of ties.

Finally, we consider some degenerate cases. Let the prior concerning both players lie on the segment $[Q, H]$ in the relevant strategy space. Both players have initially two optimal actions, $\{(1,2),\ (1,3)\}$ and $\{(2,2),\ (2,3)\}$, respectively. Thus, we need to apply the logarithmic tracing procedure. When we project the related path of Nash equilibria we derive that the path of restricted Nash equilibria moves into the regions related to (C, C). This because both optimal actions will be treated equally, i.e., both paths initially move towards the middle of $[B, C]$. Other boundary cases can be evaluated similarly.

To summarize we may again subdivide the strategy space into attraction sets related to each of the Nash equilibria. With the information collected above we derive that the equilibrium (B, B) has the attraction set with largest size.

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